

THERE IS NO UPPER BOUND FOR THE DIAMETER OF THE COMMUTING GRAPH OF A FINITE GROUP

MICHAEL GIUDICI AND CHRIS PARKER

ABSTRACT. We construct a family of finite special 2-groups which have commuting graph of increasing diameter.

1. INTRODUCTION

For a group G , the *commuting graph* $\Gamma(G)$ of G is the graph which has vertices the non-central elements of G and two distinct vertices of $\Gamma(G)$ are adjacent if and only if they commute in G . In [3], Iranmanesh and Jafarzadeh conjecture that the commuting graph of a finite group is either disconnected or has diameter bounded above by a constant independent of the group G . They support this conjecture by proving that the commuting graph of $\text{Sym}(n)$ and $\text{Alt}(n)$ is either disconnected or has diameter at most 5. The conjecture is verified by the second author for the special case of soluble groups with trivial centre in [4] where it is shown that the appropriate constant for such groups is 8. This followed earlier work of Woodcock [7] and Giudici and Pope [1]. Further support for the conjecture is provided by the work of Segev and Seitz which demonstrates that the commuting graph of a classical simple group defined over a field of order greater than 5 is either disconnected or has diameter at most 10 and at least 4 [5, Corollary (pg. 127), Theorem 8]. In addition they show the commuting graph of the exceptional Lie type groups other than $E_7(q)$ and the sporadic simple groups are disconnected [5, Theorem 6]. In [2] Hegarty and Zhelezov suggest a construction of a class of 2-groups motivated by probabilistic methods aimed at providing a counter example to the Iranmanesh and Jafarzadeh conjecture. Though as yet unsuccessful, their putative examples motivated the examples presented in this article. Their supporting calculations yielded a group with commuting graph having diameter 10, the largest known diameter in the literature. Our theorem is as follows.

Theorem 1.1. *For all positive integers d , there exists a finite special 2-group G such that the commuting graph of G has diameter greater than d .*

Theorem 1.1 proves that the Iranmanesh and Jafarzadeh conjecture is false. However, we believe that it is most probably true that the commuting graph of a finite group with trivial centre is either disconnected or has diameter bound above by a constant. We are confident enough in this judgment to formulate it as a formal conjecture:

Conjecture 1.1. *There is an absolute constant d such that if G is a finite group with trivial centre, then the commuting graph of G is either disconnected or has diameter at most d .*

We remark, that if the definition of the commuting graph of a group is revised so that the vertices of the graph are *all* the non-trivial elements of G , then our conjecture is that the modified commuting graph of a finite group G is either disconnected or has diameter bounded above by a constant independent of G .

Acknowledgement. The authors are grateful to the Banff International Research Station for supporting the Groups and Geometries conference in September 2012. The research reported in this article is a direct result of discussions initiated at that research meeting. The first author's work on this research was supported by an ARC Discovery Project.

2. THE CONSTRUCTION

Suppose that m is an integer with $m \geq 3$ and V_m and W_m are vector spaces defined over $\text{GF}(2)$ which have dimension m and $m - 2$ respectively. Assume x_1, \dots, x_m is an ordered basis for V_m and y_1, \dots, y_{m-2} is an ordered basis for W_m . Let $f_m : V_m \times V_m \rightarrow W_m$ be the bilinear map defined by the bilinear extension of the following map

$$f_m(x_i, x_j) = \begin{cases} 0 & j \in \{i, i+1\} \\ y_{j-i-1} & i+2 \leq j \leq m \\ 0 & i > j \end{cases}.$$

Because f_m is bilinear it is immediate that it is a 2-cocycle. Therefore we can define the group H_m which has underlying set $V_m \times W_m$ and multiplication defined as follows: for $(a, b), (c, d) \in H_m$,

$$(a, b) \cdot (c, d) = (a + c, f_m(a, c) + b + d).$$

Then H_m is a central extension of V_m by W_m . Furthermore, as f_m is non-zero, H_m is a nilpotent group of class 2. Note that the identity of H_m is $(0, 0)$.

We calculate $(x_i, 0)(x_i, 0) = (0, f_m(x_i, x_i)) = (0, 0)$ so that $(x_1, 0), \dots, (x_m, 0)$ are involutions and

$$\begin{aligned} [(x_i, 0), (x_j, 0)] &= (x_i, 0)(x_j, 0)(x_i, 0)(x_j, 0) \\ &= \begin{cases} (0, f_m(x_i, x_j)) = (0, y_{j-i-1}) & i+1 < j \\ (0, f_m(x_i, x_j)) = (0, y_{i-j-1}) & i > j+1 \\ (0, 0) & \text{otherwise} \end{cases}. \end{aligned}$$

The following lemma is elementary to prove.

Lemma 2.1. *Assume that $m \geq 4$. Then the following hold:*

- (i) *We have $C_{H_m}((x_1, 0)) = \langle (x_1, 0), (x_2, 0), (0, w) \mid w \in W_m \rangle$ and $C_{H_m}((x_m, 0)) = \langle (x_m, 0), (x_{m-1}, 0), (0, w) \mid w \in W_m \rangle$.*
- (ii) *$[H_m, H_m] = Z(H_m) = \{(0, w) \in H_m \mid w \in W_m\}$ has order 2^{m-2} .*
- (iii) *$X = \{(v, 0) \in H_m \mid v \in V_m\}$ is a transversal to $Z(H_m)$ in H_m .*
- (iv) *$\langle (x_1, 0), \dots, (x_{m-1}, 0) \rangle \cong H_{m-1}$.*

Proof. To see (i), first note that $C_{H_m}((x_1, 0)) \geq \langle (0, w) \mid w \in W_m \rangle$. Suppose that $(v, 0) \in C_{H_m}((x_1, 0))$ with $v \in V_m \setminus \{0\}$. Write $v = x_{i_1} + \cdots + x_{i_r}$ with $1 \leq i_1 \leq \cdots \leq i_r$. If $i_r \geq 3$, then $[(x_1, 0), (v, 0)] = (0, y_{i_1-2} + \cdots + y_{i_r-2}) \neq (0, 0)$. Thus $i_r \leq 2$ and this proves the first part of (i). The proof of the second part is similar.

Clearly

$$[H_m, H_m] = \langle (0, y_{j-i-1}) \mid 1 \leq i < j \leq m \rangle = \{(0, w) \mid w \in W_m\} \leq Z(H_m).$$

On the other hand, by (i), as $m \geq 4$,

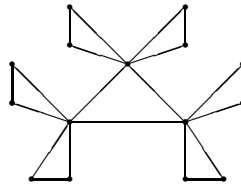
$$Z(H_m) \leq C_{H_m}((x_1, 0)) \cap C_{H_m}((x_m, 0)) = \{(0, w) \mid w \in W_m\}.$$

So (ii) holds.

Parts (iii) and (iv) are obvious. □

We now commence with the investigation of the commuting graph $\Gamma_m = \Gamma(H_m)$ of H_m . We define a subgraph Γ_m^* of Γ_m . The vertices of Γ_m^* are the non-trivial elements of the transversal $X = \{(v, 0) \in H_m \mid v \in V_m\}$ to $Z(H_m)$ and two elements of $X \setminus \{(0, 0)\}$ are joined if and only if they commute. Then Γ_m is the lexicographic product of Γ_m^* and the complete graph on $|Z(H_m)|$ vertices [6]. Thus the diameter of Γ_m is equal to the diameter of Γ_m^* . In particular, to prove Theorem 1.1, it suffices to prove that for every natural number d , there exists m such that Γ_m^* has diameter greater than d . This is now our objective. To make the notation less unwieldy we abbreviate the elements $(x_i, 0)$, $1 \leq i \leq m$, by x_i and $(0, y_i)$, $1 \leq i \leq m-2$, by y_i expecting that no significant confusion will occur. We also set $Z_m = Z(H_m)$.

We know that Γ_4^* has 15 vertices and elementary calculations yield that it has a graphical representation as follows:



The graph Γ_4^* .

Therefore Γ_4^* is connected and has diameter 3.

The proof that Γ_m is connected only uses the fact that $\dim V_m - \dim W_m \geq 2$ and the connectivity of smaller graphs.

Lemma 2.2. *For all $m \geq 4$, Γ_m is connected.*

Proof. We have already seen that Γ_4^* is connected. Hence Γ_4 is connected. Assume $m > 4$ and that Γ_{m-1} is connected.

Let $J = \langle x_1, \dots, x_{m-1} \rangle Z_m$. Then J has index 2 in H_m , $\Gamma(J)$ is a subgraph of Γ_m and $\Gamma(J) \cong \Gamma_{m-1}$ is connected. Let $a \in H_m \setminus J$. It suffices to show $C_{H_m}(a) \cap J \not\leq Z_m$. This means we should show $|C_{H_m}(a)/Z_m| \geq 4$. The commutator map $\phi : H_m/Z_m \rightarrow H'_m$ given

by $bZ \mapsto [a, b]$ is a homomorphism from $H_m/Z_m \cong V_m$ of order 2^m to $H'_m \cong W_m$ which has order 2^{m-2} and so, indeed, $|C_{H_m}(a)/Z_m| \geq 4$ and therefore Γ_m is connected. \square

Lemma 2.3. *Suppose that d is an integer such that $m > 2^{d-1}$. Assume that $w \in V(\Gamma_m^*)$ and $d(x_1, w) = d$. If x_n appears in the minimal expression for w , then $n \leq 2^{d-1} + 1$.*

Proof. We have $C_{H_m}(x_1) = \langle x_1, x_2 \rangle Z_m$ from Lemma 2.1 (i). Hence $|C_{H_m}(x_1)/Z_m| = 4$ and the vertices incident to x_1 in Γ_m^* are $x_1 + x_2$ and x_2 . Thus the highest subscript involved in vertices at distance 1 from x_1 is $2 = 2^{1-1} + 1$. So the result is true for vertices at distance 1 from x_1 . Assume that the result is true for vertices at distance k from x_1 . Let $w \in V(\Gamma_m^*)$ be such that $d(x_1, w) = k + 1$ and $u \in V(\Gamma_m^*)$ be incident to w and have distance k from x_1 . Write $w = x_{\beta_1} + \dots + x_{\beta_s}$ and $u = x_{\alpha_1} + \dots + x_{\alpha_r}$ where $\alpha_1 \leq \dots \leq \alpha_r$ and $\beta_1 \leq \dots \leq \beta_s$. Since $d(x_1, u) = k$, $\alpha_r \leq 2^{k-1} + 1$. Because $[u, w] = (0, 0)$, we have

$$\sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} [x_{\alpha_i}, x_{\beta_j}] = \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} y_{|\alpha_i - \beta_j| - 1} = (0, 0),$$

where we assume y_ℓ with $\ell \leq 0$ is $(0, 0)$. If $\beta_s \leq \alpha_r + 1 \leq 2^{k-1} + 2$, then there is nothing to prove. Hence we may assume that $\beta_s \geq \alpha_r + 2 \geq \alpha_1 + 2$. As $\sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} y_{|\alpha_i - \beta_j| - 1} = (0, 0)$, there exists $\alpha_1 \leq \alpha_t \leq \alpha_r$ and $\beta_1 \leq \beta_u \leq \beta_s$ such that

$$y_{\beta_s - \alpha_1 - 1} = \begin{cases} y_{\alpha_t - \beta_u - 1} & \alpha_t > \beta_u \\ y_{\beta_u - \alpha_t - 1} & \alpha_t < \beta_u \end{cases}.$$

Since $\beta_s \geq \beta_u$ and $\alpha_1 \leq \alpha_t$, the latter possibility is impossible. Thus $\beta_s - \alpha_1 - 1 = \alpha_t - \beta_u - 1$ which means

$$\beta_s < \beta_s + \beta_u = \alpha_1 + \alpha_t \leq 2\alpha_r \leq 2(2^{k-1} + 1).$$

Therefore $\beta_s \leq 2^k + 1$ and the result follows by induction. \square

Proof of Theorem 1.1. Lemmas 2.2 and 2.3 show that for any given integer d , there exists a positive integer m such that Γ_m is connected of diameter greater than d . \square

One final remark: computations show that for $4 \leq m \leq 16$ the diameter of Γ_m is $m - 1$.

REFERENCES

- [1] Giudici M. and Pope A. On bounding the diameter of the commuting graph of a group, arXiv:1206.3731v2, 2012.
- [2] Hegarty, P. and Zhelezov, D. Can connected commuting graphs of finite groups have arbitrarily large diameter? arXiv:1204.5456v3, 2012.
- [3] Iranmanesh A. and Jafarzadeh A. On the commuting graph associated with the symmetric and alternating groups, J. Algebra Appl., 7 (2008), 129–146.
- [4] Parker, C. The commuting graph of a soluble group. arXiv:1209.2279v, 2012.
- [5] Segev Y. and Seitz G. M. Anisotropic groups of type A_n and the commuting graph of finite simple groups. Pacific Journal of Mathematics. 202 (2002), 125–225.
- [6] Vahidi J. and Asghar Talebi, A. The commuting graphs on groups D_{2n} and Q_n . *The Journal of Mathematics and Computer Science*, Vol .1 No.2 (2010) 123–127.

[7] Woodcock, T. J. Commuting Graphs of Finite Groups. PhD thesis, University of Virginia, 2010.

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY WA 6009, AUSTRALIA

E-mail address: `michael.giudici@uwa.edu.au`

SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM B15 2TT, UNITED KINGDOM

E-mail address: `c.w.parker@bham.ac.uk`